

Limits and Continuity in Higher Dimensions

P. Sam Johnson

**National Institute of Technology Karnataka (NITK)
Surathkal, Mangalore, India**



Overview

In the lecture, we shall discuss limits and continuity for multivariable functions.

The definition of the limit of a function of two or three variables is similar to the definition of the limit of a function of a single variable but with a crucial difference, as we now see in the lecture.

Limits

If the values of $f(x, y)$ lie arbitrarily close to a fixed real number L for all points (x, y) sufficiently close to a point (x_0, y_0) we say that f approaches the limit L as (x, y) approaches (x_0, y_0) . This is similar to the informal definition for the limit of a function of a single variable.

Notice, however, that if (x_0, y_0) lies in the interior of f 's domain, (x, y) can approach (x_0, y_0) **from any direction**. For the limit to exist, the same limiting value must be obtained whatever direction of approach is taken. We illustrate this issue in several examples following the definition.

Definition 1 (Limit of a Function of Two Variables).

We say that a function $f(x, y)$ approaches the limit L as (x, y) approaches (x_0, y_0) , and write

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f ,

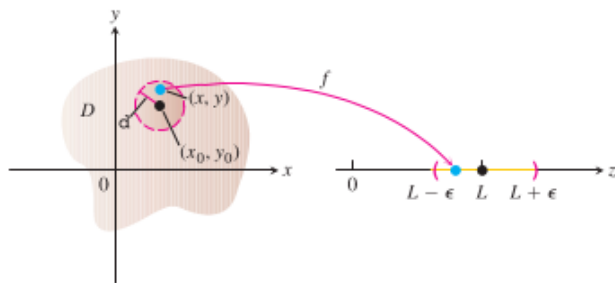
$$|f(x, y) - L| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

Limits

The definition of limit says that the distance between $f(x, y)$ and L becomes arbitrarily small whenever the distance from (x, y) to (x_0, y_0) is made sufficiently small (but not 0).

The definition of limit applies to interior points (x_0, y_0) as well as boundary points of the domain of f , although a boundary point need not lie within the domain f . The points (x, y) that approach (x_0, y_0) are always taken to be in the domain of f .

Limits



In the limit definition, δ is the radius of a disk centered at (x_0, y_0) .

For all points (x, y) within this disk, the function values $f(x, y)$ lie inside the corresponding interval $(L - \epsilon, L + \epsilon)$.

Illustration of Definition of Limit Through Figures

The following figure illustrates the definition of limit by means of an arrow diagram. If any small interval $(L - \varepsilon, L + \varepsilon)$ is given around L , then we can find a disk D_δ with center (a, b) and radius $\delta > 0$ such that f maps all the points in D_δ [except possibly (a, b)] into the interval $(L - \varepsilon, L + \varepsilon)$.

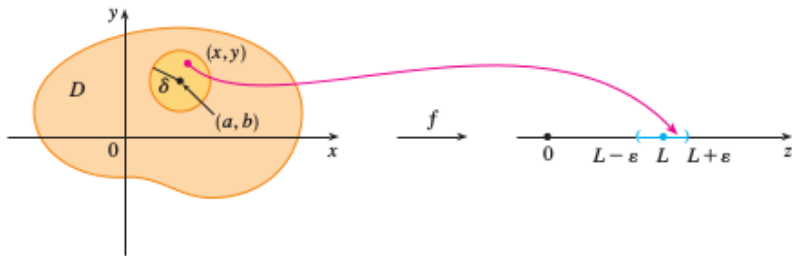
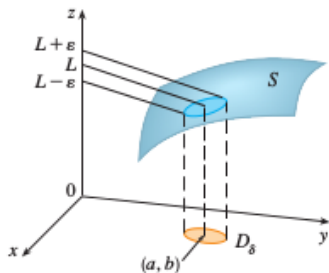


Illustration of Definition of Limit Through Figures

Another illustration of the definition of limit is given in the following figure where the surface S is the graph of f . If $\varepsilon > 0$ is given, we can find $\delta > 0$ such that if (x, y) is restricted to lie in the disk D_δ and $(x, y) \neq (a, b)$, then the corresponding part of S lies between the horizontal planes $z = L + \varepsilon$ and $z = L - \varepsilon$.



How to show mathematically that L is not the limit of $f(x, y)$ as $(x, y) \rightarrow (x_0, y_0)$?

It is enough to find **a particular value** for ε (**there exists** an $\varepsilon > 0$), such that for any $\delta > 0$ with

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta, \quad (1)$$

the image of **some** (x, y) satisfying (1) under f , that is, $f(x, y)$, will not lie in the interval

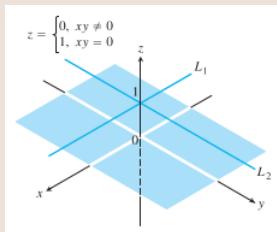
$$(L - \varepsilon, L + \varepsilon).$$

This has been explained in the following example.

How to show mathematically that L is not the limit of $f(x, y)$ as $(x, y) \rightarrow (x_0, y_0)$?

Example 2.

$$\text{Let } f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0. \end{cases}$$



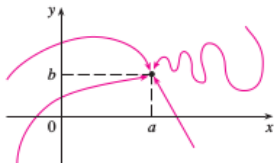
The limit of f as (x, y) approaches $(0, 0)$ does not exist because of the following reason:

For $\varepsilon = \frac{1}{2}$, take any disc D centered at $(0, 0)$ with a positive radius δ , the image of every point in the disc D will be either 0 or 1. Note that the images of some points in the disc D will not lie in the interval $(L - \varepsilon, L + \varepsilon) = (1 - \frac{1}{2}, 1 + \frac{1}{2}) = (\frac{1}{2}, \frac{3}{2})$.

Limits

For functions of a single variable, when we let x approach a , there are only two possible directions of approach, from the left or from the right. We know that if $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$, then $\lim_{x \rightarrow a} f(x)$ does not exist.

For functions of two variables the situation is not as simple because we can let (x, y) approach (a, b) from an infinite number of directions in any manner whatsoever (see the following figure) as long as (x, y) stays within the domain of f .



Limits

It can be shown, as for functions of a single variable, that

$$\lim_{(x,y) \rightarrow (x_0,y_0)} x = x_0$$

$$\lim_{(x,y) \rightarrow (x_0,y_0)} y = y_0$$

$$\lim_{(x,y) \rightarrow (x_0,y_0)} k = k \quad (\text{any number } k).$$

Limits

For example, in the first limit statement above, $f(x, y) = x$ and $L = x_0$. Using the definition of limit, suppose that $\varepsilon > 0$ is chosen. If we let δ equal this ε , we see that

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta = \varepsilon$$

implies

$$\begin{aligned} 0 < \sqrt{(x - x_0)^2} < \varepsilon \\ |x - x_0| < \varepsilon \\ |f(x, y) - x_0| < \varepsilon \end{aligned}$$

Limits

That is,

$$|f(x, y) - x_0| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

So

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = \lim_{(x,y) \rightarrow (x_0,y_0)} x = x_0.$$

It can also be shown that the limit of the sum of two functions is the sum of their limits (when they both exist), with similar results for the limits of the differences, products, constant multiples, quotients, powers and roots.

Theorem 3 (Properties of Limits of Functions of Two Variables).

The following rules hold if L , M , and k are real numbers and

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y) \rightarrow (x_0,y_0)} g(x,y) = M.$$

1. Sum Rule :
$$\lim_{(x,y) \rightarrow (x_0,y_0)} (f(x,y) + g(x,y)) = L + M$$
2. Difference Rule :
$$\lim_{(x,y) \rightarrow (x_0,y_0)} (f(x,y) - g(x,y)) = L - M$$
3. Product Rule :
$$\lim_{(x,y) \rightarrow (x_0,y_0)} (f(x,y) \cdot g(x,y)) = L \cdot M$$
4. Constant Multiple Rule :
$$\lim_{(x,y) \rightarrow (x_0,y_0)} (kf(x,y)) = kL \quad (\text{any number } k).$$

Theorem 4 (contd...).

5. Quotient Rule :
$$\lim_{(x,y) \rightarrow (x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M} \quad M \neq 0$$

6. Power Rule :
$$\lim_{(x,y) \rightarrow (x_0,y_0)} [f(x,y)]^n = L^n, \quad n \text{ is a positive integer}$$

7. Root Rule :
$$\lim_{(x,y) \rightarrow (x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n},$$

n is a positive integer, and if n is even, we assume that L > 0.

8. Power and Root Rules : *If r and s are integers with no common factors, and s ≠ 0, then*

$$\lim_{(x,y) \rightarrow (x_0,y_0)} (f(x,y))^{r/s} = L^{r/s}$$

provided L^{r/s} is a real number. (If s is even, we assume that L > 0.)

Informal Proof

If (x, y) is sufficiently close to (x_0, y_0) , then $f(x, y)$ is close to L and $g(x, y)$ is close to M (from the informal interpretation of limits).

It is then reasonable that $f(x, y) + g(x, y)$ is close to $L + M$; $f(x, y) - g(x, y)$ is close to $L - M$; $f(x, y)g(x, y)$ is close to LM ; $kf(x, y)$ is close to kL ; and that $f(x, y)/g(x, y)$ is close to L/M if $M \neq 0$.

When we apply Theorem (3) to polynomials and rational functions, we obtain the useful result that the limits of these functions as $(x, y) \rightarrow (x_0, y_0)$ can be calculated by evaluating the functions at (x_0, y_0) . The only requirement is that the rational functions be defined at (x_0, y_0) .

Example : Calculating Limits

Example 5.

Find $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$.

Solution : Since the denominator $\sqrt{x} - \sqrt{y}$ approaches 0 as $(x, y) \rightarrow (0, 0)$, we cannot use the Quotient Rule from Theorem (3). If we multiply numerator and denominator by $\sqrt{x} + \sqrt{y}$, however, we produce an equivalent fraction whose limit we can find

$$\begin{aligned}\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} &= \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 - xy)(\sqrt{x} + \sqrt{y})}{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})} \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{x(x - y)(\sqrt{x} + \sqrt{y})}{x - y} \\ &= \lim_{(x,y) \rightarrow (0,0)} x(\sqrt{x} + \sqrt{y}) \\ &= 0(\sqrt{0} + \sqrt{0}) = 0\end{aligned}$$

Limits

We can cancel the factor $(x - y)$ because the path $y = x$ (along which $x - y = 0$) is not in the domain of the function

$$\frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$$

Applying $\varepsilon - \delta$ definition

When we use the definition of a limit to show that a particular limit exists, we usually employ certain key or basic inequalities such as:

$$\blacksquare |x| < \sqrt{x^2 + y^2}$$

$$\blacksquare |y| < \sqrt{x^2 + y^2}$$

$$\blacksquare \frac{x}{x+1} < 1$$

$$\blacksquare \frac{x^2}{x^2+y^2} < 1$$

$$\blacksquare |x - a| = \sqrt{(x - a)^2} \leq \sqrt{(x - a)^2 + (y - b)^2}$$

$$\blacksquare |y - b| = \sqrt{(y - b)^2} \leq \sqrt{(x - a)^2 + (y - b)^2}$$

Example : Applying the Limit Definition

Example 6.

Find $\lim_{(x,y) \rightarrow (0,0)} \frac{4xy^2}{x^2+y^2}$ if it exists.

Solution : We first observe that along the line $x = 0$, the function always has value 0 when $y \neq 0$. Likewise, along the line $y = 0$, the function has value 0 provided $x \neq 0$. So if the limit does exist as (x, y) approaches $(0,0)$, the value of the limit must be 0. To see if this is true, we apply the definition of limit.

Let $\varepsilon > 0$ be given, but arbitrary. We want to find a $\delta > 0$ such that

$$\left| \frac{4xy^2}{x^2 + y^2} - 0 \right| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta$$

or

$$\frac{4|x|y^2}{x^2 + y^2} < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta.$$

Example (contd...)

Since $y^2 \leq x^2 + y^2$ we have that

$$\frac{4|x|y^2}{x^2 + y^2} \leq 4|x| = 4\sqrt{x^2} \leq 4\sqrt{x^2 + y^2}.$$

so if we choose $\delta = \varepsilon/4$ and let $0 < \sqrt{x^2 + y^2} < \delta$, we get

$$\left| \frac{4xy^2}{x^2 + y^2} - 0 \right| \leq 4\sqrt{x^2 + y^2} < 4\delta = 4\left(\frac{\varepsilon}{4}\right) = \varepsilon.$$

It follows from the definition that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{4xy^2}{x^2 + y^2} = 0.$$

Example

Example 7.

If $f(x, y) = \frac{y}{x}$, does $\lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y)$ exist?

Solution : The domain of f does not include the y -axis, so we do not consider any points (x, y) where $x = 0$ in the approach toward the origin $(0, 0)$.

Along the x -axis, the value of the function is $f(x, 0) = 0$ for all $x \neq 0$. So if the limit does exist as $(x, y) \rightarrow (0, 0)$, the value of the limit must be $L = 0$.

Solution (contd...)

On the other hand, along the line $y = x$, the value of the function is $f(x, x) = x/x = 1$ for all $x \neq 0$. That is, the function f approaches the value 1 along the line $y = x$. This means that for every disk of radius δ centered at $(0, 0)$, the disk will contain points $(x, 0)$ on the x -axis where the value of the function is 0, and also points (x, x) along the line $y = x$ where the value of the function is 1.

So no matter how small we choose δ as the radius of the disk, there will be points within the disk for which the function values differ by 1. Therefore, the limit cannot exist because we can take ε to be any number less than 1 in the limit definition and deny that $L = 0$ or 1, or any other real number. The limit does not exist because we have different limiting values along different paths approaching the point $(0, 0)$.

As with functions of a single variable, continuity is defined in terms of limits.

Definition 8 (Continuous Function of Two Variables).

A function $f(x, y)$ is continuous at the point (x_0, y_0) if

1. f is defined at (x_0, y_0) ,
2. $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y)$ exists,
3. $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = f(x_0, y_0)$.

A function is continuous if it is continuous at every point of its domain.

Continuity

As with the definition of limit, the definition of continuity applies at boundary points as well as interior points of the domain of f . The only requirement is that the point (x, y) remain in the domain at all times.

As you may have guessed, one of the consequences of Theorem (3) is that algebraic combinations of continuous functions are continuous at every point at which all the functions involved are defined. This means that sums, differences, products, constant multiples, quotients, and powers of continuous functions are continuous where defined. In particular, polynomials and rational functions of two variable are continuous at every point at which they are defined.

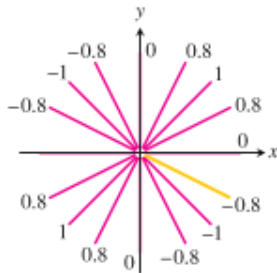
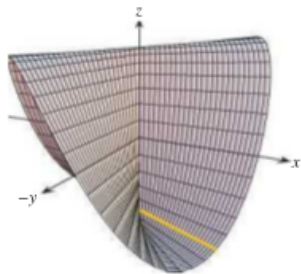
Example : A Function with a Single Point of Discontinuity

Example 9.

Show that

$$f(x, y) = \begin{cases} \frac{2xy}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous at every point except the origin.



Solution

The function f is continuous at any point $(x, y) \neq (0, 0)$ because its values are then given by a rational function of x and y .

At $(0, 0)$, the value of f is defined, but f , we claim, has no limit as $(x, y) \rightarrow (0, 0)$. The reason is that different paths of approach to the origin can lead to different results, as we now see.

For every value of m , the function f has a constant value on the punctured line $y = mx, x \neq 0$, because

$$f(x, y) \Big|_{y=mx} = \frac{2xy}{x^2 + y^2} \Big|_{y=mx} = \frac{2x(mx)}{x^2 + (mx)^2} = \frac{2mx^2}{x^2 + m^2x^2} = \frac{2m}{1 + m^2}.$$

Solution (contd...)

Therefore, f has this number as its limit as (x, y) approaches $(0, 0)$ along the line:

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=mx}} f(x, y) = \lim_{(x,y) \rightarrow (0,0)} \left[f(x, y) \Big|_{y=mx} \right] = \frac{2m}{1+m^2}.$$

This limit changes with each value of the slope m .

Therefore there is no single number we may call the limit of f as (x, y) approaches the origin.

The limit fails to exist, and the function is not continuous.

Examples (7) and (9) illustrate an important point about limits of functions of two variables (or even more variables, for that matter). For a limit to exist at a point, the limit must be the same along every approach path.

This result is analogous to the single-variable case where both the left- and right-sided limits had to have the same value; therefore, for functions of two or more variables, if we ever find paths with different limits, we know the function has no limit at the point they approach.

Two-Path Test for Nonexistence of a Limit : If a function $f(x, y)$ has different limits along two different paths as (x, y) approaches (x_0, y_0) , then $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y)$ does not exist.

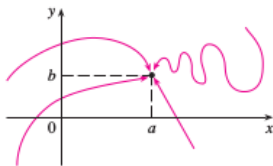
Limits

For functions of a single variable, when we let x approach a , there are only two possible directions of approach, from the left or from the right. We know that if

$$\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$$

then $\lim_{x \rightarrow a} f(x)$ does not exist.

For functions of two variables the situation is not as simple because we can let (x, y) approach (a, b) from an infinite number of directions in any manner whatsoever (see the following figure) as long as (x, y) stays within the domain of f .



Limits

The definition of limit says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (x_0, y_0) sufficiently small (but not 0).

The definition refers only to the distance between (x, y) and (x_0, y_0) . It does not refer to the direction of approach. Therefore, if the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (x_0, y_0) . Thus if we can find two different paths of approach along which the function $f(x, y)$ has different limits, then it follows that

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y)$$

does not exist.

Two-Path Test for Nonexistence of a Limit

If a function $f(x, y)$ has different limits along two different paths as (x, y) approaches (x_0, y_0) , then $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y)$ does not exist.

In other words, if $f(x, y) \rightarrow L_1$ as $(x, y) \rightarrow (x_0, y_0)$ along a path C_1 and $f(x, y) \rightarrow L_2$ as $(x, y) \rightarrow (x_0, y_0)$ along a path C_2 , where $L_1 \neq L_2$, then

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y)$$

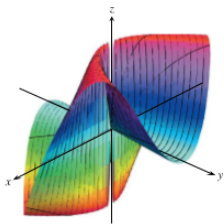
does not exist.

Example : Applying the Two-Path Test

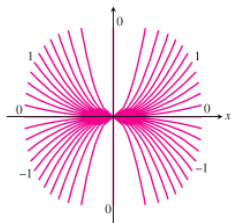
Example 10.

Show that the function $f(x, y) = \frac{2x^2y}{x^4 + y^2}$ has no limit as (x, y) approaches $(0, 0)$.

The graph of $f(x, y) = \frac{2x^2y}{x^4 + y^2}$.



Along each path $y = kx^2$ the value of f is constant, but varies with k .



Solution

The limit cannot be found by direct substitution, which gives the form $0/0$. We examine the values of f along curves that end at $(0, 0)$. Along the curve $y = kx^2$, $x \neq 0$, the function has the constant value

$$f(x, y)|_{y=kx^2} = \frac{2x^2y}{x^4 + y^2}|_{y=kx^2} = \frac{2x^2(kx^2)}{x^4 + (kx^2)^2} = \frac{2kx^4}{x^4 + k^2x^4} = \frac{2k}{1 + k^2}.$$

Therefore,

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=kx^2}} f(x, y) = \lim_{(x,y) \rightarrow (0,0)} [f(x, y)|_{y=kx^2}] = \frac{2k}{1 + k^2}.$$

This limit varies with the path of approach. If (x, y) approaches $(0, 0)$ along the parabola $y = x^2$, for instance, $k = 1$ and the limit is 1 . If (x, y) approaches $(0, 0)$ along the x -axis, $k = 0$ and the limit is 0 . By the two-path test, f has no limit as (x, y) approaches $(0, 0)$.

Applying the Two-Path Test : An observation

It can be shown that the function $f(x, y) = \frac{2x^2y}{x^4 + y^2}$ has limit 0 along every path $y = mx$.

Having the same limit along all straight lines approaching (x_0, y_0) does not imply a limit exists at (x_0, y_0) .

Continuity of Composites

Whenever it is correctly defined, the composite of continuous functions is also continuous. The only requirement is that each function be continuous where it is applied. The proof, omitted here, is similar to that for functions of a single variable.

Definition 11.

If f is continuous at (x_0, y_0) and g is a single-variable function continuous at $f(x_0, y_0)$, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is continuous at (x_0, y_0) .

For example, the composite functions

$$e^{x-y}, \quad \cos \frac{xy}{x^2 + 1}, \quad \ln(1 + x^2 y^2)$$

are continuous at every point (x, y) .

Functions of More Than Two Variables

The definitions of limit and continuity for functions of two variables and the conclusions about limits and continuity for sums, products, quotients, powers, and composites all extend to functions of three or more variables. Functions like

$$\ln(x + y + z) \quad \text{and} \quad \frac{y \sin z}{x - 1}$$

are continuous throughout their domains, and limits like

$$\lim_{P \rightarrow (1,0,-1)} \frac{e^{(x+z)}}{z^2 + \cos \sqrt{xy}} = \frac{e^{1-1}}{(-1)^2 + \cos 0} = \frac{1}{2},$$

where P denotes the point (x, y, z) , may be found by direct substitution.

Exercise 12.

Find the limits of the following.

$$(a) \quad \lim_{(x,y) \rightarrow (0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}$$

$$(b) \quad \lim_{(x,y) \rightarrow (0,0)} \cos \frac{x^2 + y^3}{x + y + 1}$$

$$(c) \quad \lim_{(x,y) \rightarrow (1/27, \pi^3)} \cos \sqrt[3]{xy}$$

$$(d) \quad \lim_{(x,y) \rightarrow (1, \pi/6)} \frac{x \sin y}{x^2 + 1}$$

Solution for Exercise 12

$$(a) \quad \lim_{(x,y) \rightarrow (0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2} = \frac{3(0)^2 - 0^2 + 5}{0^2 + 0^2 + 2} = \frac{5}{2}.$$

$$(b) \quad \lim_{(x,y) \rightarrow (0,0)} \cos\left(\frac{x^2 + y^2}{x + y + 1}\right) = \cos\left(\frac{0^2 + 0^2}{0 + 0 + 1}\right) = \cos 0 = 1.$$

$$(c) \quad \lim_{(x,y) \rightarrow (1/27, \pi^3)} \cos \sqrt[3]{xy} = \cos \sqrt[3]{\left(\frac{1}{27}\right) \pi^3} = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}.$$

$$(d) \quad \lim_{(x,y) \rightarrow (1, \pi/6)} \frac{x \sin y}{x^2 + 1} = \frac{1 - \sin\left(\frac{\pi}{6}\right)}{1^2 + 1} = \frac{1/2}{2} = \frac{1}{4}.$$

Limits of Quotients

Exercise 13.

Find the limits by rewriting the fractions first.

$$(a) \lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq y}} \frac{x^2 - 2xy + y^2}{x - y}$$

$$(b) \lim_{\substack{(x,y) \rightarrow (2,-4) \\ y \neq -4, x \neq x^2}} \frac{y + 4}{x^2y - xy + 4x^2 - 4x}$$

$$(c) \lim_{\substack{(x,y) \rightarrow (2,0) \\ 2x - y \neq 4}} \frac{\sqrt{2x - y} - 2}{2x - y - 4}$$

$$(d) \lim_{\substack{(x,y) \rightarrow (4,3) \\ x \neq y+1}} \frac{\sqrt{x} - \sqrt{y+1}}{x - y - 1}$$

Solution for (a) and (b) in Exercise 13

$$(a) \lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq y}} \frac{x^2 - 2xy + y^2}{x - y} = \lim_{(x,y) \rightarrow (1,1)} \frac{(x - y)^2}{x - y} = (1 - 1) = 0$$

$$(b) \lim_{\substack{(x,y) \rightarrow (2,-4) \\ y \neq -4, x \neq x^2}} \frac{y + 4}{x^2y - xy + 4x^2 - 4x} = \lim_{\substack{(x,y) \rightarrow (2,-4) \\ y \neq -4, x \neq x^2}} \frac{y + 4}{x(x - 1)(y + 4)} =$$

$$\lim_{\substack{(x,y) \rightarrow (2,-4) \\ y \neq -4, x \neq x^2}} \frac{1}{x(x - 1)} = \frac{1}{2(2 - 1)} = \frac{1}{2}$$

Solution for (c) and (d) in Exercise 13

$$(c) \quad \lim_{\substack{(x,y) \rightarrow (2,0) \\ 2x-y \neq 4}} \frac{\sqrt{2x-y}-2}{2x-y-4} = \lim_{\substack{(x,y) \rightarrow (2,0) \\ 2x-y \neq 4}} \frac{\sqrt{2x-y}-2}{(\sqrt{2x-y}+2)(\sqrt{2x-y}-2)} =$$
$$\lim_{(x,y) \rightarrow (2,0)} \frac{1}{\sqrt{2x-y}+2} = \frac{1}{\sqrt{(2)(2)-0}+2} = \frac{1}{2+2} = \frac{1}{4}$$

$$(d) \quad \lim_{\substack{(x,y) \rightarrow (4,3) \\ x \neq y+1}} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1} =$$
$$\lim_{\substack{(x,y) \rightarrow (4,3) \\ x \neq y+1}} \frac{\sqrt{x}-\sqrt{y+1}}{(\sqrt{x}+\sqrt{y+1})(\sqrt{x}-\sqrt{y+1})} =$$
$$\lim_{(x,y) \rightarrow (4,3)} \frac{1}{\sqrt{x}+\sqrt{y+1}} = \frac{1}{\sqrt{4}+\sqrt{3+1}} = \frac{1}{2+2} = \frac{1}{4}$$

Limits with Three Variables

Exercise 14.

Find the limits.

(a) $\lim_{P \rightarrow (3,3,0)} (\sin^2 x + \cos^2 y + \sec^2 z)$

(b) $\lim_{P \rightarrow (\pi,0,3)} ze^{-2y} \cos 2x$

(c) $\lim_{P \rightarrow (2,-3,6)} \ln \sqrt{x^2 + y^2 + z^2}$

Solution for Exercise 14

- (a) $\lim_{P \rightarrow (3,3,0)} (\sin^2 x + \cos^2 y + \sec^2 z) = (\sin^2 3 + \cos^2 3) + \sec^2 0 = 1 + 1^2 = 2$
- (b) $\lim_{P \rightarrow (\pi, 0, 3)} ze^{2y} \cos 2x = 3e^{2(0)} \cos 2\pi = (3)(1)(1) = 3$
- (c) $\lim_{P \rightarrow (2, -3, 6)} \ln \sqrt{x^2 + y^2 + z^2} = \ln \sqrt{2^2 + (-3)^2 + 6^2} = \ln \sqrt{49} = \ln 7$

Continuity in the plane

Exercise 15.

At what points (x, y) in the plane are the following functions continuous?

(a) $f(x, y) = \sin(x + y)$

(b) $g(x, y) = \sin \frac{1}{xy}$

(c) $g(x, y) = \frac{x^2 + y^2}{x^2 - 3x + 2}$

Solution for Exercise 15

- (a) All (x, y) .
- (b) All (x, y) except where $x = 0$ or $y = 0$.
- (c) All (x, y) so that $x^2 - 3x + 2 \neq 0 \Rightarrow (x - 2)(x - 1) \neq 0$.
Hence $x \neq 2$ and $x \neq 1$.

Exercise 16.

At what points (x, y, z) in space are the following functions continuous?

(a) $f(x, y, z) = x^2 + y^2 - 2z^2$

(b) $h(x, y, z) = xy \sin \frac{1}{z}$

(c) $h(x, y, z) = \frac{1}{|y|+|z|}$

Solution for Exercise 16

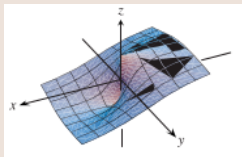
- (a) All (x, y, z) .
- (b) All (x, y, z) with $z \neq 0$.
- (c) All (x, y, z) except $(x, 0, 0)$.

No Limit at a point

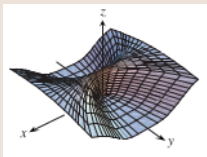
Exercise 17.

By considering different paths of approach, show that the following functions have no limit as $(x, y) \rightarrow (0, 0)$.

(a) $f(x, y) = -\frac{x}{\sqrt{x^2+y^2}}$



(b) $f(x, y) = \frac{x^4}{x^4+y^2}$



Solution for Exercise 17

$$(a) \lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=x \\ x>0}} -\frac{x}{\sqrt{x^2+y}} = \lim_{x \rightarrow 0} -\frac{x}{\sqrt{x^2+x^2}} = \lim_{x \rightarrow 0} -\frac{x}{\sqrt{2}|x|} =$$

$$\lim_{x \rightarrow 0} -\frac{x}{\sqrt{2}x} = \lim_{x \rightarrow 0} -\frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}};$$

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=x \\ x<0}} -\frac{x}{\sqrt{x^2+y^2}} = \lim_{x \rightarrow 0} -\frac{x}{\sqrt{2}|x|} = \lim_{x \rightarrow 0} -\frac{x}{\sqrt{2}(-x)} = \frac{1}{\sqrt{2}}.$$

$$(b) \lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=0}} \frac{x^4}{x^2+y^2} = \lim_{x \rightarrow 0} \frac{x^4}{x^2+0^2} = 1;$$

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=x^2}} \frac{x^4}{x^2+y^2} = \lim_{x \rightarrow 0} \frac{x^4}{x^2+(x^2)^2} = \lim_{x \rightarrow 0} \frac{x^4}{2x^2} = \frac{1}{2}.$$

No Limit at a point

Exercise 18.

By considering different paths of approach, show that the following functions have no limit as $(x, y) \rightarrow (0, 0)$.

(a) $f(x, y) = \frac{x^4 - y^2}{x^4 + y^2}$

(b) $h(x, y) = \frac{x^2 y}{x^4 + y^2}$

Solution for Exercise 18

$$(a) \lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=kx^2}} \frac{x^4 - y^2}{x^4 + y^2} = \lim_{x \rightarrow 0} \frac{x^4 - (kx^2)^2}{x^2 + (kx^2)^2} = \lim_{x \rightarrow 0} \frac{x^4 - k^2x^4}{x^4 + k^2x^4} = \frac{1 - k^2}{1 + k^2}.$$

Hence we get different limits for different values of k . Thus the limit does not exist.

$$(b) \lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=kx^2}} \frac{x^2y}{x^4 + y^2} =, \lim_{x \rightarrow 0} \frac{kx^4}{x^4 + k^2x^4} = \frac{k}{1 + k^2}.$$

Hence we get different limits for different values of k . Thus the limit does not exist.

Exercises 19.

1. If $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = L$, must f be defined at (x_0, y_0) ? Give reasons for your answer.
2. If $f(x_0, y_0) = 3$, what can you say about

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y)$$

if f is continuous at (x_0, y_0) ? If f is not continuous at (x_0, y_0) ? Give reasons for your answer.

Solution for Exercise 19

(a) f may not be defined at the point. See Example 6 in the notes.

$\lim_{(x,y) \rightarrow (0,0)} \frac{4xy^2}{x^2+y^2}$ exists but the function is not defined at $(0,0)$.

(b) If f is continuous at (x_0, y_0) , then $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y)$ must equal $f(x_0, y_0) = 3$. If f is not continuous at (x_0, y_0) , the limit could have any value different from 3, and need not even exist.

The Sandwich Theorem

The Sandwich Theorem for functions of two variables states that if $g(x, y) \leq f(x, y) \leq h(x, y)$ for all $(x, y) \neq (x_0, y_0)$ in a disk centered at (x_0, y_0) and if g and h have the same finite limit L as $(x, y) \rightarrow (x_0, y_0)$, then

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = L$$

Use this result to support your answers to the questions in next two exercises (Exercises 20 and 21).

Exercises 20.

1. Does knowing that $1 - \frac{x^2y^2}{3} < \frac{\tan^{-1} xy}{xy} < 1$ tell you anything about

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\tan^{-1} xy}{xy} ?$$

Give reasons for your answer.

2. Does knowing that $2|xy| - \frac{x^2y^2}{6} < 4 - 4 \cos \sqrt{|xy|} < 2|xy|$ tell you anything about

$$\lim_{(x,y) \rightarrow (0,0)} \frac{4 - 4 \cos \sqrt{|xy|}}{|xy|} ?$$

Give reasons for your answer.

Solution for Exercise 20

(a) $\lim_{(x,y) \rightarrow (0,0)} \left(1 - \frac{x^2 y^2}{3}\right) = 1$ and $\lim_{(x,y) \rightarrow (0,0)} 1 = 1$.

Hence $\lim_{(x,y) \rightarrow (0,0)} \frac{\tan^{-1} xy}{xy} = 1$, by the Sandwich Theorem.

(b) If $xy > 0$, $\lim_{(x,y) \rightarrow (0,0)} \frac{2|xy| - (\frac{x^2 y^2}{6})}{|xy|} = \lim_{(x,y) \rightarrow (0,0)} \frac{2xy - (\frac{x^2 y^2}{6})}{xy} = \lim_{(x,y) \rightarrow (0,0)} \left(2 - \frac{xy}{6}\right) =$

2 and $\lim_{(x,y) \rightarrow (0,0)} \frac{2|xy|}{|xy|} = \lim_{(x,y) \rightarrow (0,0)} 2 = 2$;

if $xy < 0$, $\lim_{(x,y) \rightarrow (0,0)} \frac{2|xy| - (\frac{x^2 y^2}{6})}{|xy|} = \lim_{(x,y) \rightarrow (0,0)} \frac{-2xy - (\frac{x^2 y^2}{6})}{-xy} = \lim_{(x,y) \rightarrow (0,0)} \left(2 + \frac{xy}{6}\right) =$

2 and $\lim_{(x,y) \rightarrow (0,0)} \frac{2|xy|}{|xy|} = 2$.

Hence $\lim_{(x,y) \rightarrow (0,0)} \frac{4 - 4 \cos \sqrt{|xy|}}{|xy|} = 2$, by the Sandwich Theorem.

Exercises 21.

1. Does knowing that $|\sin(1/x)| \leq 1$ tell you anything about

$$\lim_{(x,y) \rightarrow (0,0)} y \sin \frac{1}{x} ?$$

Give reasons for your answer.

2. Does knowing that $|\cos(1/y)| \leq 1$ tell you anything about

$$\lim_{(x,y) \rightarrow (0,0)} x \cos \frac{1}{y} ?$$

Give reasons for your answer.

Solution for Exercise 21

(a) The limit is 0 since $|\sin(\frac{1}{x})| \leq 1 \Rightarrow -1 \leq \sin(\frac{1}{x}) \leq 1$.

Hence $-y \leq y \sin(\frac{1}{x}) \leq y$ for $y \geq 0$, and

$-y \geq y \sin(\frac{1}{x}) \geq y$ for $y \leq 0$.

As $(x, y) \rightarrow (0, 0)$, both $-y$ and y approach 0.

Thus $y \sin(\frac{1}{x}) \rightarrow 0$, by the Sandwich Theorem.

(b) The limit is 0 since $|\cos(\frac{1}{y})| \leq 1 \Rightarrow -1 \leq \cos(\frac{1}{y}) \leq 1$.

Hence $-x \leq x \cos(\frac{1}{y}) \leq x$ for $x \geq 0$, and

$-x \geq x \cos(\frac{1}{y}) \geq x$ for $x \leq 0$.

As $(x, y) \rightarrow (0, 0)$, both $-x$ and x approach 0.

Thus $x \cos(\frac{1}{y}) \rightarrow 0$, by the Sandwich Theorem.

Exercises 22.

- (a) Consider the function $f(x, y) = \frac{y}{x}$. To find the limit of $f(x, y)$ as $(x, y) \rightarrow (0, 0)$, substitute $m = \tan \theta$ into the formula

$$f(x, y) \Big|_{y=mx} = \frac{2m}{1+m^2}$$

and simplify the result to show how the value of f varies with the line's angle of inclination.

- (b) Use the formula you obtained in part (a) to show that the limit of f as $(x, y) \rightarrow (0, 0)$ along the line $y = mx$ varies from -1 to 1 depending on the angle of approach.

Solution for Exercise 22

- (a) Along the line $y = mx$, $f(x, y) = \frac{2m}{1+m^2} = \frac{2 \tan \theta}{1+\tan^2 \theta} = \sin 2\theta$.
The value of $f(x, y) = \sin 2\theta$ varies with θ , which is the line's angle of inclination.
- (b) Since if $f(x, y)|_{y=mx} = \sin 2\theta$ and since $-1 \leq \sin 2\theta \leq 1$ for every θ ,

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y)$$

varies from -1 to 1 along $y = mx$.

Exercise 23 (Continuous extension).

Define $f(0,0)$ in a way that extends

$$f(x,y) = xy \frac{x^2 - y^2}{x^2 + y^2}$$

to be continuous at the origin.

Solution for Exercise 23

$$|xy(x^2 - y^2)| = |xy||x^2 - y^2| \leq |x||y||x^2 + y^2| = \sqrt{x^2}\sqrt{y^2}|x^2 + y^2| \leq \sqrt{x^2 + y^2}\sqrt{x^2 + y^2}|x^2 + y^2| = (x^2 + y^2)^2.$$

So,

$$\left| \frac{xy(x^2 - y^2)}{x^2 + y^2} \right| \leq \frac{(x^2 + y^2)^2}{x^2 + y^2} = x^2 + y^2 \Rightarrow -(x^2 + y^2) \leq \frac{xy(x^2 - y^2)}{x^2 + y^2} \leq (x^2 + y^2).$$

Hence $\lim_{(x,y) \rightarrow (0,0)} \left(xy \frac{x^2 - y^2}{x^2 + y^2} \right) = 0$ by the Sandwich Theorem, since

$$\lim_{(x,y) \rightarrow (0,0)} \pm(x^2 + y^2) = 0; \text{ thus, define } f(0,0) = 0.$$

Iterated Limits

The limits of the type

$$\lim_{x \rightarrow a} \lim_{y \rightarrow b} f(x, y) \quad \text{and} \quad \lim_{y \rightarrow b} \lim_{x \rightarrow a} f(x, y)$$

are called **iterated limits**.

Exercise

Exercise 24.

Let

$$f(x, y) = \frac{x^2 y^2}{x^2 y^2 + (x^2 - y^2)^2}$$

for $(x, y) \neq (0, 0)$. Show that the iterated limits

$$\lim_{y \rightarrow 0} \left(\lim_{x \rightarrow 0} f(x, y) \right) \quad \text{and} \quad \lim_{x \rightarrow 0} \left(\lim_{y \rightarrow 0} f(x, y) \right)$$

exist, but

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$$

does not exist.

Solution for Exercise 24

$$\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} \frac{x^2 y^2}{x^2 y^2 + (x^2 - y^2)^2} = 0 = \lim_{x \rightarrow 0} \lim_{y \rightarrow 0} \frac{x^2 y^2}{x^2 y^2 + (x^2 - y^2)^2}.$$

Along the line $y = mx$,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x^2 (mx)^2}{x^2 (mx)^2 + (x - mx)^2} &= \lim_{x \rightarrow 0} \frac{m^2 x^4}{m^2 x^4 + x^2 (1 - m)^2} \\ &= \lim_{x \rightarrow 0} \frac{m^2}{m^2 + (1 - m)^2 x^{-2}} \\ &= \begin{cases} 0 & m \neq 1 \\ 1 & m = 1. \end{cases} \end{aligned}$$

Hence

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{x^2 y^2 + (x^2 - y^2)^2}$$

does not exist.

Iterated Limits

There are functions f such that

$$\lim_{(x,y) \rightarrow (0,0)} f(x,y)$$

exists, but the iterated limit

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} f(x,y)$$

does not exist.

For instance, consider the function $f : \mathbb{R}^2 \setminus \{(0,0)\} \rightarrow \mathbb{R}$ defined by

$$f(x,y) = x \sin \frac{1}{x} \sin \frac{1}{y}.$$

Changing to Polar Coordinates

If you cannot make any headway with

$$\lim_{(x,y) \rightarrow (0,0)} f(x,y)$$

in rectangular coordinates, try changing to polar coordinates.

Substitute $x = r \cos \theta$, $y = r \sin \theta$, and investigate the limit of the resulting expression as $r \rightarrow 0$.

In other words, try to decide whether there exists a number L satisfying the following criterion :

Given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all r and θ ,

$$|r| < \delta \quad \Rightarrow \quad |f(r, \theta) - L| < \varepsilon. \quad (2)$$

If such an L exists, then $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{r \rightarrow 0} f(r, \theta) = L$.

Changing to Polar Coordinates

For instance,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{r^3 \cos^3 \theta}{r^2} = \lim_{r \rightarrow 0} r \cos^3 \theta = 0.$$

To verify the last of these equalities, we need to show that Equation (2) is satisfied with $f(r, \theta) = r \cos^3 \theta$ and $L = 0$. That is, we need to show that given any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all r and θ ,

$$|r| < \delta \quad \Rightarrow \quad |r \cos^3 \theta - 0| < \varepsilon.$$

Since

$$|r \cos^3 \theta| = |r| |\cos^3 \theta| \leq |r| \cdot 1 = |r|,$$

the implication holds for all r and θ if we take $\delta = \varepsilon$.

Caution!

If $\lim_{r \rightarrow 0} f(r, \theta) = L$, **then L may not be** $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$.

To prove that “our guess L ” could be the limit, we should prove by $\varepsilon - \delta$ definition :

given any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all r and θ ,

$$|r| < \delta \quad \Rightarrow \quad |f(r, \theta) - L| < \varepsilon.$$

This has been explained in the following example. It illustrates that the limit may exist along every straight line (or ray) $\theta = \text{constant}$ and yet fail to exist in the broader sense.

Changing to Polar Coordinates

Example 25.

In polar coordinates, $f(x, y) = (2x^2y)/(x^4 + y^2)$ becomes

$$f(r \cos \theta, r \sin \theta) = \frac{r \cos \theta \sin 2\theta}{r^2 \cos^4 \theta + \sin^2 \theta}$$

for $r \neq 0$. If we hold θ constant and let $r \rightarrow 0$, the limit is 0. On the path $y = x^2$, however, we have $r \sin \theta = r^2 \cos^2 \theta$ and

$$\begin{aligned} f(r \cos \theta, r \sin \theta) &= \frac{r \cos \theta \sin 2\theta}{r^2 \cos^4 \theta + (r \cos^2 \theta)^2} \\ &= \frac{2r \cos^2 \theta \sin \theta}{2r^2 \cos^4 \theta} = \frac{r \sin \theta}{r^2 \cos^2 \theta} = 1. \end{aligned}$$

We have already proved in Example 10 that the function

$f(x, y) = \frac{2x^2y}{x^4 + y^2}$ has no limit as (x, y) approaches $(0, 0)$.

Changing to Polar Coordinates

Example 26.

In contrast,

$$\frac{x^2}{x^2 + y^2} = \frac{r^2 \cos^2 \theta}{r^2} = \cos^2 \theta$$

takes on all values from 0 to 1 regardless of how small $|r|$ is, so that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2 + y^2}$$

does not exist.

After changing the function to “Polar Coordinates,” if we get the value of $\lim_{r \rightarrow 0} f(r, \theta)$ depending on θ , we can say that $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ does not exist.

Exercises 27.

1. Find the limit of f as $(x, y) \rightarrow (0, 0)$ or show that the limit does not exist.

(a) $f(x, y) = \frac{x^3 - xy^2}{x^2 + y^2}$

(b) $f(x, y) = \tan^{-1}\left(\frac{|x| + |y|}{x^2 + y^2}\right)$

2. In the following exercises, define $f(0, 0)$ in a way that extends f to be continuous at the origin.

(a) $f(x, y) = \ln\left(\frac{3x^2 - x^2y^2 + 3y^2}{x^2 + y^2}\right)$

(b) $f(x, y) = \frac{3x^2y}{x^2 + y^2}$

Solution for Exercise (1.) in 27

$$(a) \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - xy^2}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{r^3 \cos^3 \theta - (r \cos \theta)(r^3 \sin^2 \theta)}{r^3 \cos^2 \theta + r^3 \sin^2 \theta} = \lim_{r \rightarrow 0} \frac{r(\cos^3 \theta - \cos \theta \sin^2 \theta)}{1} = 0.$$

$$(b) \lim_{(x,y) \rightarrow (0,0)} \tan^{-1} \left[\frac{|x| + |y|}{x^2 + y^2} \right] = \lim_{r \rightarrow 0} \tan^{-1} \left[\frac{|r \cos \theta| + |r \sin \theta|}{|r^2|} \right] =$$

$$\lim_{r \rightarrow 0} \tan^{-1} \left[\frac{|r|(|\cos \theta| + |\sin \theta|)}{r^2} \right];$$

$$\text{if } r \rightarrow 0^+, \text{ then } \lim_{r \rightarrow 0} \tan^{-1} \left[\frac{|r|(|\cos \theta| + |\sin \theta|)}{r^2} \right] = \lim_{r \rightarrow 0} \tan^{-1} \left[\frac{|\cos \theta| + |\sin \theta|}{r} \right] = \frac{\pi}{2};$$

$$\text{if } r \rightarrow 0^-, \text{ then } \lim_{r \rightarrow 0} \tan^{-1} \left[\frac{|r|(|\cos \theta| + |\sin \theta|)}{r^2} \right] = \lim_{r \rightarrow 0} \tan^{-1} \left[\frac{|\cos \theta| + |\sin \theta|}{-r} \right] = \frac{\pi}{2}.$$

Hence the limit is $\frac{\pi}{2}$.

Solution for Exercise (2.) in 27

$$(a) \quad \lim_{(x,y) \rightarrow (0,0)} \ln \left(\frac{3x^2 - x^2y^2 + 3y^2}{x^2 + y^2} \right) = \lim_{r \rightarrow 0} \ln \left(\frac{3r^2 \cos^2 \theta - r^2 \cos^2 \theta \sin^2 \theta + 3r^2 \sin^2 \theta}{r^2} \right) =$$

$$\lim_{r \rightarrow 0} \ln(3 - r^2 \cos^2 \theta \sin^2 \theta) = \ln 3.$$

Define $f(0,0) = \ln 3$.

$$(b) \quad \lim_{(x,y) \rightarrow (0,0)} \frac{3xy^2}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{(3r \cos \theta)(r^2 \sin^2 \theta)}{r^2} = \lim_{r \rightarrow 0} 3r \cos \theta \sin^2 \theta = 0.$$

Define $f(0,0) = 0$.

Using the $\delta - \varepsilon$ Definition

Exercises 28.

Each of the following exercises gives a function $f(x, y)$ and a positive number ε . In each exercise, show that there exists a $\delta > 0$ such that for all (x, y) ,

$$\sqrt{x^2 + y^2} < \delta \quad \Rightarrow \quad |f(x, y) - f(0, 0)| < \varepsilon.$$

(a) $f(x, y) = x^2 + y^2, \quad \varepsilon = 0.01$

(b) $f(x, y) = (x + y)/(x^2 + 1), \quad \varepsilon = 0.01$

Solution for Exercise 28

- (a) Let $\delta = 0.1$. Then $\sqrt{x^2 + y^2} < \delta \Rightarrow \sqrt{x^2 + y^2} < 0.1 \Rightarrow x^2 + y^2 < 0.01 \Rightarrow |x^2 + y^2 - 0| < 0.01 \Rightarrow |f(x, y) - f(0, 0)| < 0.01 = \varepsilon$.
- (b) Let $\delta = 0.005$. Then $|x| < \delta$ and $|y| < \delta \Rightarrow |f(x, y) - f(0, 0)| = \left| \frac{x+y}{x^2+1} - 0 \right| = \left| \frac{x+y}{x^2+1} \right| \leq |x+y| < |x| + |y| < 0.005 + 0.005 = 0.01 = \varepsilon$.

Exercises 29.

Each of the following exercises gives a function $f(x, y, z)$ and a positive number ε . In each exercise, show that there exists a $\delta > 0$ such that for all (x, y, z) ,

$$\sqrt{x^2 + y^2 + z^2} < \delta \quad \Rightarrow \quad |f(x, y, z) - f(0, 0, 0)| < \varepsilon.$$

(a) $f(x, y, z) = x^2 + y^2 + z^2, \quad \varepsilon = 0.015$

(b) $f(x, y, z) = xyz, \quad \varepsilon = 0.0008$

(c) $f(x, y, z) = \tan^2 x + \tan^2 y + \tan^2 z, \quad \varepsilon = 0.03.$

Solution for Exercise 29

(a) Let $\delta = \sqrt{0.015}$. Then

$$\sqrt{x^2 + y^2 + z^2} < \delta \Rightarrow |f(x, y, z) - f(0, 0, 0)| = |x^2 + y^2 + z^2 - 0| = |x^2 + y^2 + z^2| = (\sqrt{x^2 + y^2 + z^2})^2 < (\sqrt{0.015})^2 = 0.015 = \varepsilon.$$

(b) Let $\delta = 0.2$. Then $|x| < \delta$, $|y| < \delta$, and $|z| < \delta$.

$$\text{Hence } |f(x, y, z) - f(0, 0, 0)| = |xyz - 0| = |xyz| = |x||y||z| < (0.2)^3 = 0.008 = \varepsilon.$$

(c) Let $\delta = \tan^{-1}(0.1)$. Then $|x| < \delta$, $|y| < \delta$, and $|z| < \delta$.

$$\text{Hence } |f(x, y, z) - f(0, 0, 0)| = |\tan^2 x + \tan^2 y + \tan^2 z| \leq |\tan^2 x| + |\tan^2 y| + |\tan^2 z| = \tan^2 x + \tan^2 y + \tan^2 z < \tan^2 \delta + \tan^2 \delta + \tan^2 \delta = 0.01 + 0.01 + 0.01 = 0.03 = \varepsilon.$$

Exercises 30.

1. Show that

$$f(x, y, z) = x + y - z$$

is continuous at every point (x_0, y_0, z_0) .

2. Show that

$$f(x, y, z) = x^2 + y^2 + z^2$$

is continuous at the origin.

Solution for Exercise 30

$$(a) \quad \lim_{(x,y,z) \rightarrow (x_0,y_0,z_0)} f(x,y,z) = \lim_{(x,y,z) \rightarrow (x_0,y_0,z_0)} (x + y - z) = x_0 + y_0 - z_0 = f(x_0, y_0, z_0).$$

Hence f is continuous at every (x_0, y_0, z_0) .

$$(b) \quad \lim_{(x,y,z) \rightarrow (x_0,y_0,z_0)} f(x,y,z) = \lim_{(x,y,z) \rightarrow (x_0,y_0,z_0)} (x^2 + y^2 + z^2) = x_0^2 + y_0^2 + z_0^2 = f(x_0, y_0, z_0).$$

Hence f is continuous at every point (x_0, y_0, z_0) .

Question 31.

Given $\varepsilon = 0.015$, which of the following δ is/are suitable for all (x, y, z) with $\sqrt{x^2 + y^2 + z^2} < \delta \implies |f(x, y, z) - f(0, 0, 0)| < \varepsilon$, where

$$f(x, y, z) = \frac{x + y + z}{x^2 + y^2 + z^2 + 2}?$$

- (A) 0.01
- (B) 0.05
- (C) 0.005
- (D) 0.02
- (E) None of the other options.

The correct answers are
0.01 ; 0.005.

Question 32.

Let $f(x, y) = \frac{\sin(2xy) - \sin 2}{xy - 1}$. Let A be the set over which f is not defined. Let g be defined over \mathbb{R}^2 such that $g(x, y) = f(x, y)$, wherever f is defined. Then which of the following is/are true?

- (A) If g is continuous at $(1, 1)$, then $g(1, 1)$ must be equal to $2 \cos 2$.
- (B) If g is continuous at $(1, 1)$, then $g(1, 1)$ must be equal to 1.
- (C) It is not possible to define g over A in such a way that g is continuous throughout \mathbb{R}^2 .
- (D) It is possible to define g over A in such a way that g is continuous throughout \mathbb{R}^2 .
- (E) None of the other options.

The correct answers are

If g is continuous at $(1, 1)$, then $g(1, 1)$ must be equal to $2 \cos 2$.

It is possible to define g over A in such a way that g is continuous throughout \mathbb{R}^2 .

Question 33.

Consider the function f defined by $f(x, y) = \frac{\tan^{-1}(xy)}{4xy}$. Then

$\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ equals:

(If you think that answer is a fraction, then write it in lowest possible terms. That is, if you think that the answer is $5/2$, then do not write $10/4$, 2.5 , or any other equivalent form. Similarly, if answer is $-8/3$, then do not write $-16/6$, -2.66 , or any other equivalent form.

*If you think that the limit does not exist, then type: limit does not exist
Avoid putting a full stop (.) at the end of your answer.)*

The correct answer is $1/4$.

0.25 is a wrong answer, as per instructions given in the question.

Students should have a habit of reading instructions also.

Old Questions

Question 34.

If $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{\sqrt{x^2 + y^2}} = 0$, then according to $\varepsilon - \delta$ definition of the limit, for a given $\varepsilon \in (0, \frac{2}{3})$, we can select one or more δ satisfying

- (A) $0 < \delta \leq \varepsilon^{\frac{2}{3}}$.
- (B) $0 < \delta \leq \varepsilon^{\frac{1}{3}}$.
- (C) $0 < \delta \leq \varepsilon^{\frac{1}{2}}$.
- (D) $0 < \delta \leq \varepsilon^{\frac{1}{6}}$.
- (E) None of the other options.

The correct answers are
 $0 < \delta \leq \varepsilon^{\frac{2}{3}}$; $0 < \delta \leq \varepsilon^{\frac{1}{2}}$.

Old Questions

Question 35.

If $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^3}{\sqrt{x^2 + y^2}} = 0$, then according to $\varepsilon - \delta$ definition of the limit, for a given $\varepsilon \in (0, \frac{1}{2})$, we can select one or more δ satisfying

- (A) $0 < \delta \leq \varepsilon^{\frac{1}{5}}$.
- (B) $0 < \delta \leq \varepsilon^{\frac{1}{3}}$.
- (C) $0 < \delta \leq \varepsilon^{\frac{1}{2}}$.
- (D) $0 < \delta \leq \varepsilon^{\frac{1}{6}}$.
- (E) None of the other options.

The correct answers are
 $0 < \delta \leq \varepsilon^{\frac{1}{3}}$; $0 < \delta \leq \varepsilon^{\frac{1}{2}}$.

Question 36.

For the given function $f(x, y) = \frac{x+y}{3+\cos x}$ and $\varepsilon = 0.008$, a positive number δ satisfying the $\varepsilon - \delta$ definition for continuity at $(0, 0)$ is/are

- (A) 0.08
- (B) 0.04
- (C) 0.008
- (D) 0.004
- (E) None of the other options.

The correct answers are
0.008 ; 0.004.

Question 37.

The function $f(x, y) = \begin{cases} 0, & \text{if } xy - 18x - 4y + 72 = 0, \\ 1, & \text{otherwise} \end{cases}$

is not continuous at

- (A) $(4, 0)$
- (B) $(0, 18)$
- (C) $(4, 18)$
- (D) $(18, 4)$
- (E) $(18, 0)$
- (F) $(0, 4)$

The correct answers are
 $(4, 0)$; $(0, 18)$; $(4, 18)$.

Question 38.

$$\text{For the given function } f(x, y) = \begin{cases} \frac{\sin(x + y)}{|x| + |y|}, & |x| + |y| \neq 0, \\ 0, & |x| + |y| = 0 \end{cases}$$

- (A) f is continuous at the origin.
- (B) f is discontinuous at the origin.
- (C) $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ fails to exist.
- (D) $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$.
- (E) None of the other options.

The correct answers are

f is discontinuous at the origin ; $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ fails to exist.

Question 39.

$\lim_{(x,y) \rightarrow (0,0)} \frac{(x^2+y^2)^{3.5}}{x^4+y^4}$ is

- (A) 0
- (B) *path independent.*
- (C) 1
- (D) ∞
- (E) *None of the other options.*

The correct answers are
0 ; path independent.

Question 40.

$\lim_{(x,y) \rightarrow (0,0)} \frac{(x^2+y^2)^{1.1}}{x^4+y^4}$ is

- (A) 0
- (B) *path independent*
- (C) 1
- (D) ∞
- (E) *None of the other options.*

The correct answers are
path independent ; ∞

Question 41.

Let $g(x, y, z) = 2(|xy| + |z|)$, and let $f(t) = \log t^2$. Also consider $A = \{(x, y, z) : x^2 + y^2 + z^2 \geq 2\}$. Then which of the following is/are true about $f \circ g$?

- (A) $f \circ g$ is continuous over A .
- (B) $f \circ g$ is not continuous over A .
- (C) $f \circ g$ is not continuous over A^c .
- (D) $f \circ g$ is continuous over A^c .
- (E) None of the other options.

The correct answers are

$f \circ g$ is not continuous over A ; $f \circ g$ is not continuous over A^c .

Question 42.

Which of the following statements is/are false?

(A) If $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = 0$ and $\lim_{(x,y) \rightarrow (a,b)} g(x,y)$ exist, then

$$\lim_{(x,y) \rightarrow (a,b)} \frac{f(x,y)}{g(x,y)} = 0.$$

(B) If $f(x,y)$ is differentiable at (x_0, y_0) , then $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist.

(C)
$$\lim_{(x,y) \rightarrow (\infty, 2)} \frac{2x^2 - y}{x^3 + (y - 1)^2} = 0.$$

(D) If $f(x,y)$ is continuous at the origin and $f(0,0) = 0$, then $|f(x,y)| < \sqrt{x^2 + y^2}$ is true in some neighbourhood of the origin.

The correct answers are

A & D

Question 43.

Let $f(x, y, z) = \frac{x + y + z}{x^2 + y^2 + z^2 + 1}$ and $\varepsilon = 0.027$. A suitable $\delta > 0$ satisfying the definition of continuity at $(0, 0, 0)$ is .

Answer : 0.009

Old Questions

Question 44.

Which of the following statements is/are true?

(A) The slope of the tangent to the intersection of the plane $y = 1$ and the hemisphere $z = \sqrt{4 - x^2 - y^2}$ at the point $(1, 1, \sqrt{2})$ is

$$\left(\frac{\partial z}{\partial y}\right)_{(1,1)}.$$

(B) If $f(x, y) = \begin{cases} \frac{xy^3}{x+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$, then $f_x(1, 1) = \frac{1}{4}$.

(C) $\lim_{(x,y) \rightarrow (0,1)} \frac{2x^2y}{(x+1)(y-1)^2} = 2$.

(D) If $f(x, y)$ is continuous at (x_0, y_0) , then $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist.

Answer : B

Question 45.

Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + y^3}{x - y}$ does not exist.

Question 46.

Prove or disprove that the function f defined below is differentiable at $(0, 0)$.

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}; & (x, y) \neq (0, 0) \\ 0; & (x, y) = (0, 0) \end{cases}$$

Old Questions

Given $f(x, y) = \begin{cases} xy^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Then

Select one or more:

- a. f is differentiable at $(0, 0)$.
- b. $f_x(0, b)$ does not exist, but $f_y(0, b)$ exist, where $b \neq 0$.
- c. f is not differentiable at $(0, 0)$.
- d. None of these
- e. Both $f_x(0, b)$ and $f_y(0, b)$ do not exist, where $b \neq 0$.

The correct answers are: f is differentiable at $(0, 0)$.

, $f_x(0, b)$ does not exist, but $f_y(0, b)$ exist, where $b \neq 0$.

Old Questions

Let $f(x, y) = \sqrt{|xy|}$ and $g(x, y) = e^{(x^2+y^2)} \cos(2x + \sqrt{5}y + \frac{\pi}{2})$. Let \hat{u} be the unit vector along which $g(x, y)$ decreases most rapidly at $(0, 0)$. Which of the following is/are true?

Select one or more:

a. $\hat{u} = \frac{2i+\sqrt{5}j}{3}$.

b. $D_{\hat{u}}f(0, 0) = \frac{\sqrt{2\sqrt{5}}}{3}$.

c. $\hat{u} = \frac{-\sqrt{5}i+2j}{3}$.

d. $D_{\hat{u}}f(0, 0)$ does not exist.

The correct answers are: $\hat{u} = \frac{2i+\sqrt{5}j}{3}$.

, $D_{\hat{u}}f(0, 0)$ does not exist.

Old Questions

The domain of $f(x, y) = \frac{y}{\ln(x-2)}$ is

Select one or more:

- a. $\{(x, y) : x \in (3, \infty), y \in \mathbb{R}\}$
- b. $\{(x, y) : x \in (2, 3) \cup (3, \infty), y \in \mathbb{R}\}$
- c. $\{(x, y) : x \in (2, \infty), y \in \mathbb{R}\}$
- d. $\{(x, y) : y \in (2, 3) \cup (3, \infty), x \in \mathbb{R}\}$

The correct answer is: $\{(x, y) : x \in (2, 3) \cup (3, \infty), y \in \mathbb{R}\}$

Old Questions

- Q.1. The function $f(x, y) = \frac{2xy^4}{x^2+y^8}$ approaches 1 as (x, y) approaches $(0, 0)$ along the path
- (a) $y^2 = x$ (b) $y^3 = x$ (c) $y^4 = x$ (d) $y^5 = x$

Solution: (c)

Old Questions

Q.4. Which of the following statements is/are FALSE?

- (a) Given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x, y) - L| \leq \epsilon$ whenever $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$ if and only if $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = L$.
- (b) Given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x, y) - L| > \epsilon$ for some point in $\{(x, y) : 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\}$ if and only if $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) \neq L$.
- (c) Given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x, y) - g(x, y)| < \epsilon$ whenever $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$ if and only if $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = \lim_{(x, y) \rightarrow (x_0, y_0)} g(x, y)$.
- (d) For some $\epsilon > 0$, there exists no $\delta > 0$ such that $|f(x, y) - L| < \epsilon$ whenever $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$ if and only if $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) \neq L$.

Solution: (b), (c) and (d)

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